SPRING 2024: BONUS PROBLEM 2

Bonus Problem 2. For up to five bonus points, use the formula for expanding determinants along a row to prove the row properties (i)-(v) for **3 × 3 matrices** as listed below.

(i) If A' is obtained form A by multiplying a row (or columns) of A times $\lambda \in F$, then $|A'| = \lambda \cdot |A|$.
(ii) If A' is obtained from A by interchanging two rows (or tow columns), then $|A'| = -|A|$.
(iii) If a row (or column) of A consists entirely of 0s, then $|A| = 0$.
(iv) If two rows (or columns) of A are the same, then $|A| = 0$.

Solution. Set $A := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Then $|A| = aei - afh + bfg - bdi + cdh - ceg$.

For (i), suppose $A' = \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{pmatrix}$. Then

$|A'| = (\lambda a)ei - (\lambda a)fh + (\lambda b)fg - (\lambda b)di + (\lambda c)dh - (\lambda c)eg = \lambda(aei - afh + bfg - bdi + cdh - ceg) = \lambda|A|$.

For (ii), suppose $A' = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$. Then $|A'| = dbi - dch + ecg - eai + fah - fbg = -|A|$.

For (iii), suppose $A' = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}$. Then $|A'| = ae0 - af0 + bf0 - bd0 + cd0 - ce0 = 0$.

For (iv), suppose $A' = \begin{pmatrix} a & b & c \\ d & e & f \\ d & e & f \end{pmatrix}$, Then $A' = aef - afe + bfd - bdf + cde - ced = 0$.